PHYSICS 2DL – SPRING 2010

MODERN PHYSICS LABORATORY

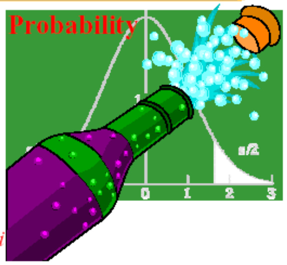
Monday April 19, 2010

Prof. Brian Keating

Homework

- Problems listed on 2DL Spring 2010 -Web Site.
- All HW problems are found in Taylor
- Hand-in HW to TA in Lab

Averaging Data



- Random Errors can be reduced by repeated measurements.
- The best estimate of the true value of a measured quantity is the average (mean).

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- We can also estimate the RMS error from the set of measurements.

$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

- We can then compute the error on the mean which decreases with the number of measurements.

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$$

If σ_x is 1 mm, how many times must I measure to get a 0.2 mm error on the mean?

Example of Error Propagation

Suppose that $q = x+y+z$. We wish to determine q by measuring x, y, and z. The error on q, (δq), can be calculated from the errors on the measured quantities.

$q = x+y+z$

$\delta q = \delta x + \delta y + \delta z$ If δx is an error estimate, we don't know its sign.

$\delta q = |\delta x| + |\delta y| + |\delta z|$ Worst case, if all have same sign.

$\sigma_q = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2}$ Independent, random errors.

σ = *RMS* = standard deviation

What is an RMS error?

Another Example of Error Propagation

$q = xy$

Definition (area): **assume the correct values are *x, y* and *q*, but, we measure *(x+δx)* and *(y+δy)*. We will then compute *(q+δq)* from our measurements.**

$$q + \delta q = (x + \delta x)(y + \delta y) = xy + x\delta y + y\delta x$$

compute

$$\delta q = x\delta y + y\delta x = \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial x}\delta x$$

Subtract *q* from both sides of the above equation and neglect *δxδy*. Notice partial derivatives.

In estimating the errors, we don't know the sign of *δx* and *δy*. Sometimes the error contributions will cancel, sometimes add. We can compute errors two ways:

- **maximum possible error** $\delta q = \left|\frac{\partial q}{\partial x}\delta x\right| + \left|\frac{\partial q}{\partial y}\delta y\right|$
- **RMS error**

Root Mean Square (RMS) Errors

$$\delta q = \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial x}\delta x$$

Compute error on $q(x,y)$ from errors one x and y. This assumes you actually know the sign and magnitude of the errors.

Define root mean square error estimate

$$\sigma_x \equiv \sqrt{\left\langle \left(x_{meas.} - x_{true}\right)^2 \right\rangle}$$

Average over many measurements.

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial q}{\partial y}\sigma_y\right)^2}$$

Propagation of errors using the RMS. <u>We will mainly use this method</u>.

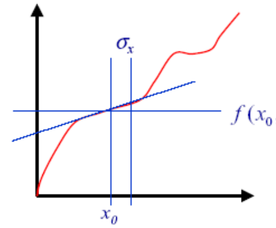
What do we do when the partial derivative is negative?

σ = (sigma) = RMS error = Standard Deviation

Why Use Partial Derivatives

1) It worked for the $q = xy$ case.

2) It makes sense graphically.



$$f(x,y,z) = f(x_0,y_0,z_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0,z_0} \sigma_x$$

3) Its really the first order Taylor expansion of a function.

$$f(x,y,z) = f(x_0,y_0,z_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0,z_0} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0,z_0} (y - y_0) + \left.\frac{\partial f}{\partial z}\right|_{x_0,y_0,z_0} (z - z_0) + \ldots$$

Fractional Errors are Sometimes Useful

For products like $q=xy$, we can add the fractional errors on the measurements to get the fractional error on the result.

$$\frac{\sigma_q}{q} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Simple Derivation

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial q}{\partial y}\sigma_y\right)^2}$$

$$\frac{\sigma_q}{q} = \frac{1}{xy}\sqrt{(y\sigma_x)^2 + (x\sigma_y)^2} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

This also works for quotients like $q = \frac{x}{y}$

For $V = \frac{4}{3}\pi r^3$, the fractional error is $\frac{\sigma_V}{V} = \frac{1}{V}\frac{\partial V}{\partial r}\sigma_r = \frac{4\pi r^2}{\frac{4}{3}\pi r^3}\sigma_r = 3\frac{\sigma_r}{r}$

Example of Error Propagation

Find the fractional error on $q(x,y)=\frac{x^n}{y}$

$$\sigma_q=\sqrt{\left(\frac{\partial q}{\partial x}\sigma_x\right)^2+\left(\frac{\partial q}{\partial y}\sigma_y\right)^2}$$ **Our basic formula.**

$$\frac{\sigma_q}{q}=\frac{y}{x^n}\sqrt{\left(\frac{nx^{n-1}}{y}\sigma_x\right)^2+\left(-\frac{x^n}{y^2}\sigma_y\right)^2}=\sqrt{\left(n\frac{\sigma_x}{x}\right)^2+\left(\frac{\sigma_y}{y}\right)^2}$$

This is the technique you'll need to memorize and use often.

Ch4: Analysis of Random Uncertainties

$x = x_1, x_2, x_3, x_4, \ldots, x_N$ **uncertainty of** x_{N+1}

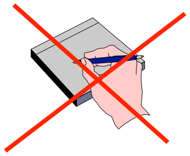
$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_N}{N} = \frac{\sum x_i}{N}$$

mean

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

standard deviation

$$x_{best}$$



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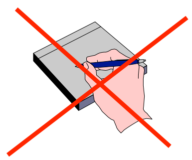
mean

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

standard deviation

$$x_{best} \pm \delta x$$



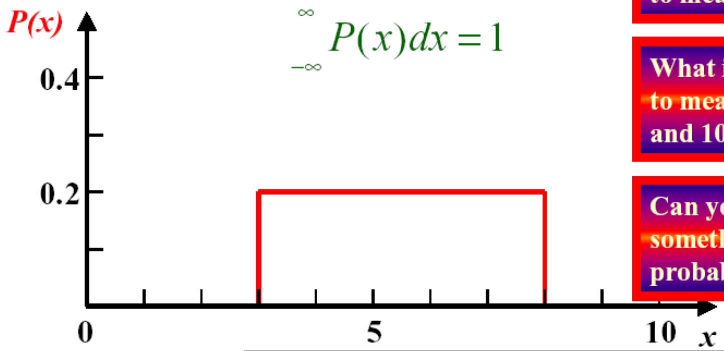
$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

standard deviation of the mean

Probability Distributions

- Assume the true value of x is 5.5 m. We make repeated measurements of x with an "error" of 2.5 m.
- What do we expect the distribution of measurements to look like? This depends on the probability distribution.
- This is one possible example.

$$\int_{-\infty}^{\infty} P(x)\,dx = 1$$

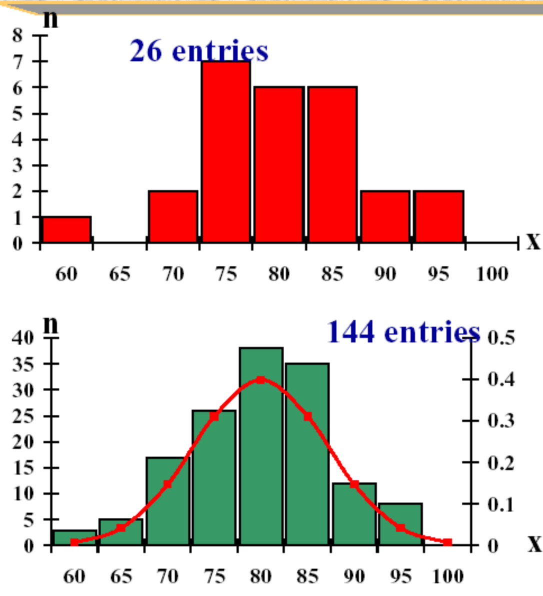


What is the probability to measure x=5?

What is the probability to measure x between 6 and 10?

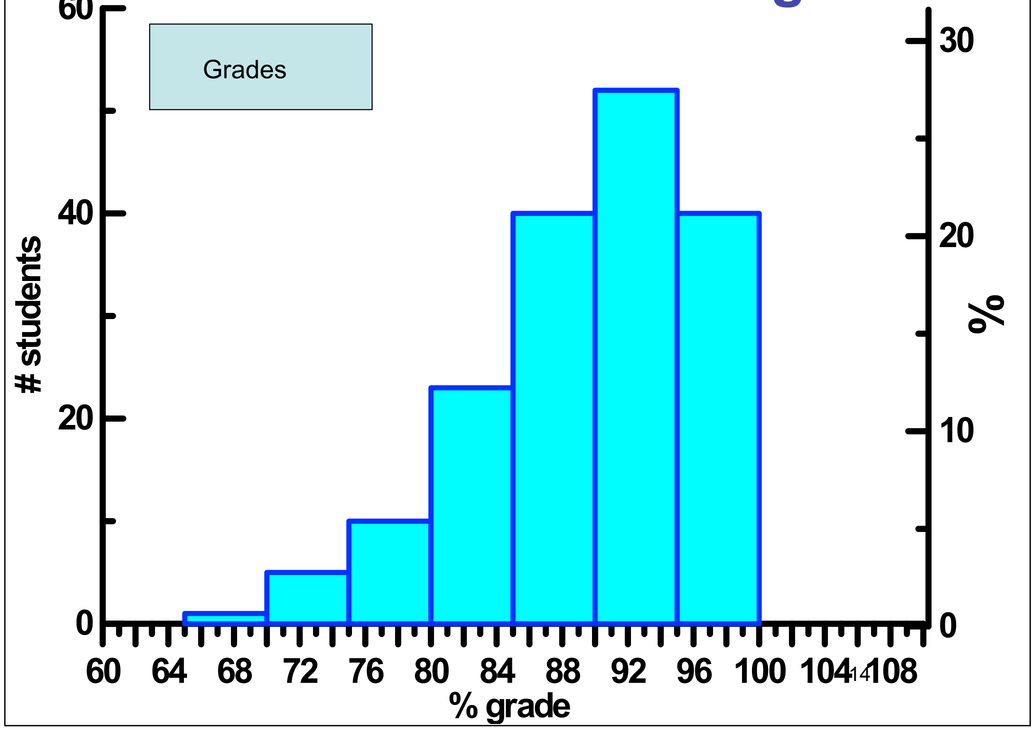
Can you think of something with a flat probability distribution?

Histograms of Measurements

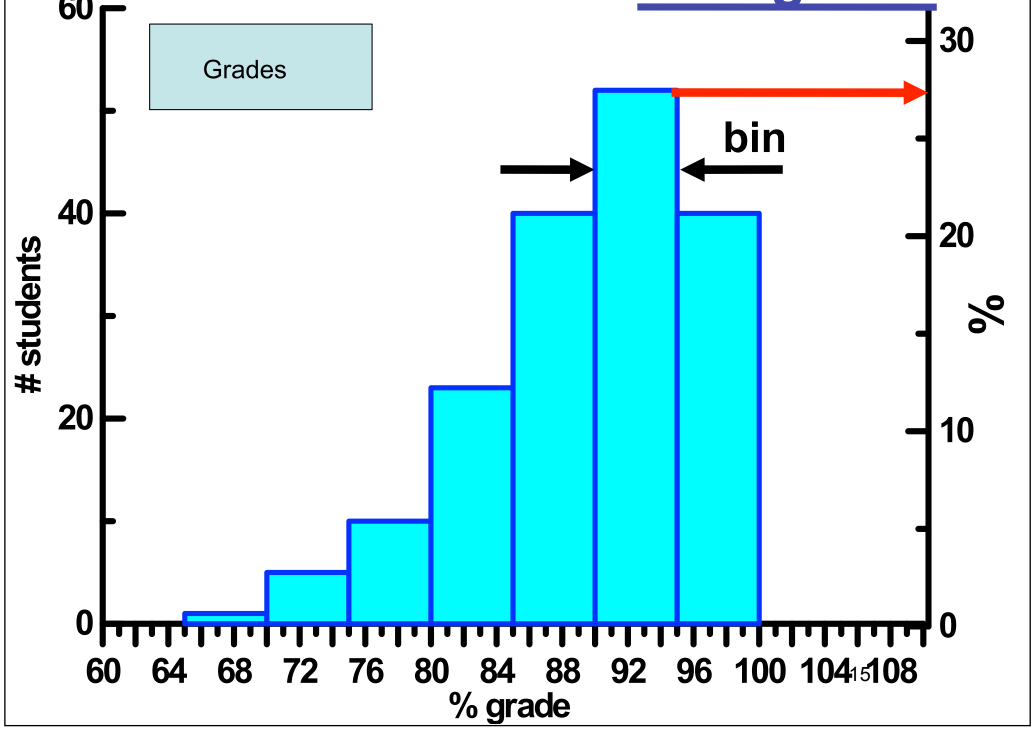


- Divide ***x*** axis into bins of equal size.
- Plot the number of entries in each bin.
- Expect statistical fluctuations for small numbers of measurements.
- For large numbers of measurements, the distribution should approach ***P(x)***.
- Usually ***P(x)*** is close to a normal distribution.

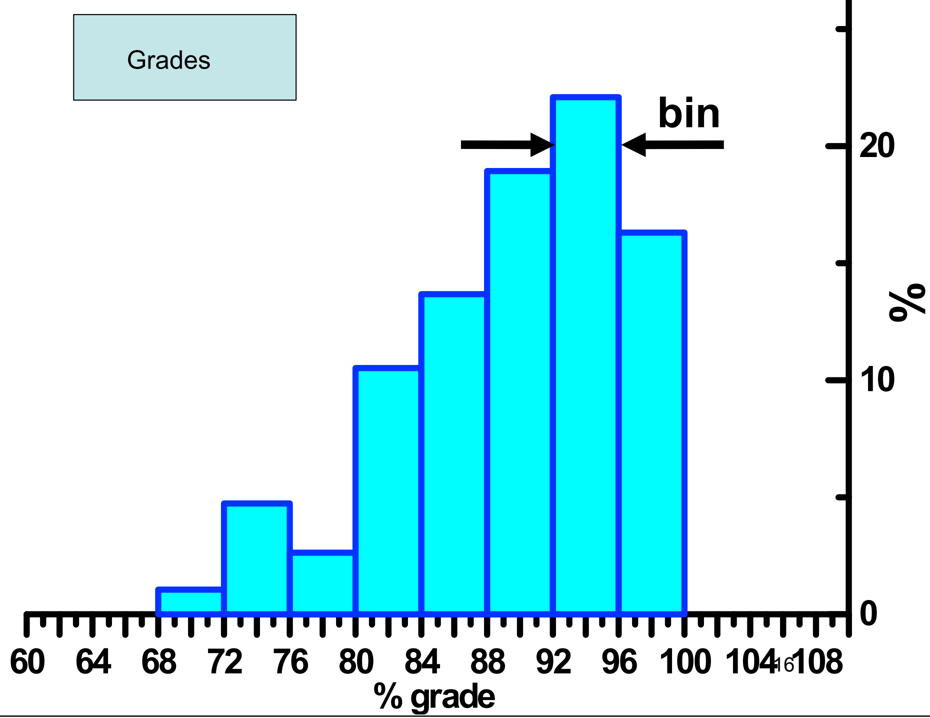
Distributions and histograms



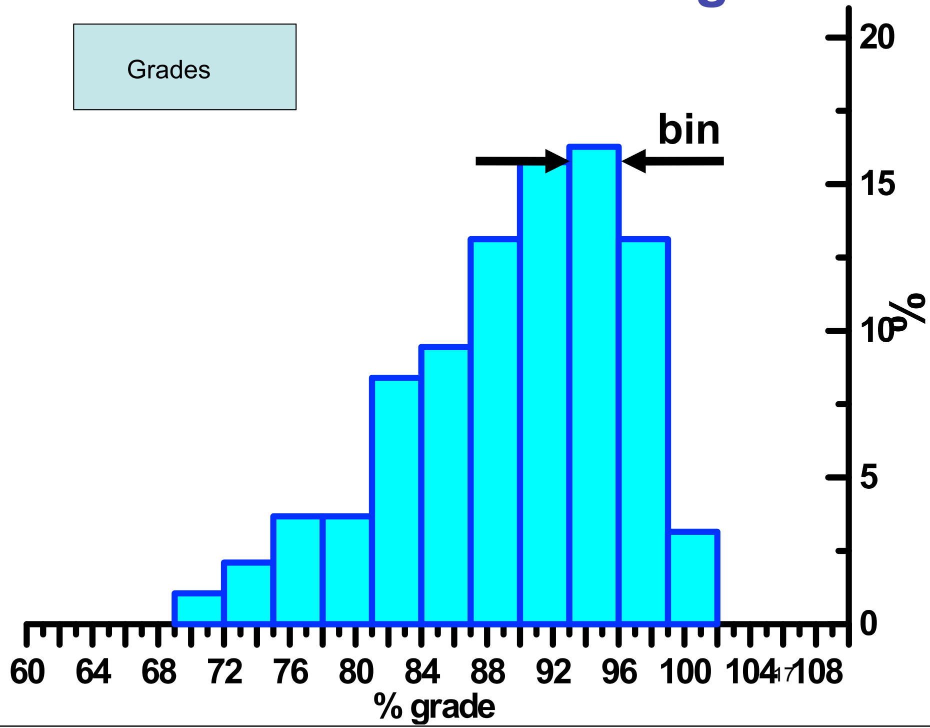
Distributions and histograms



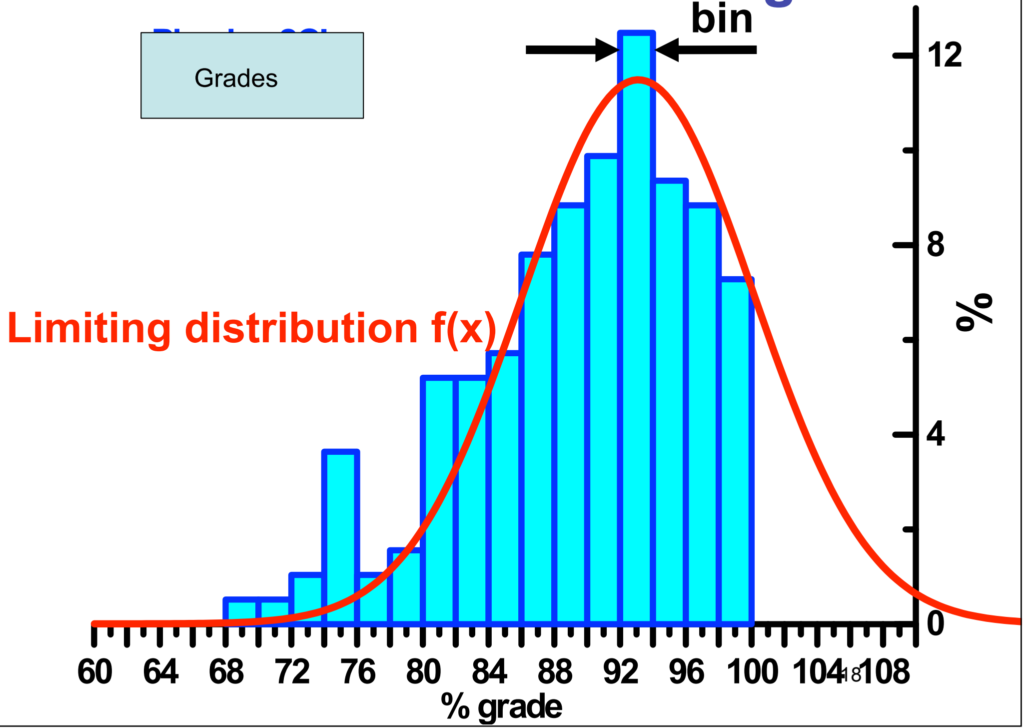
Distributions and histograms



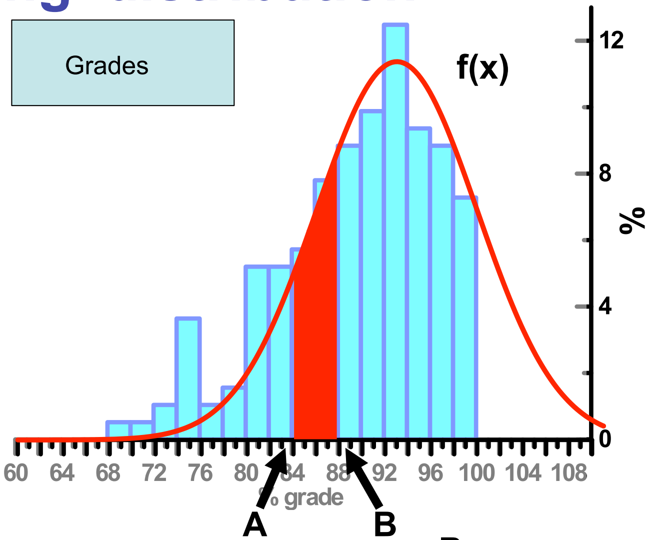
Distributions and histograms



Distributions and histograms

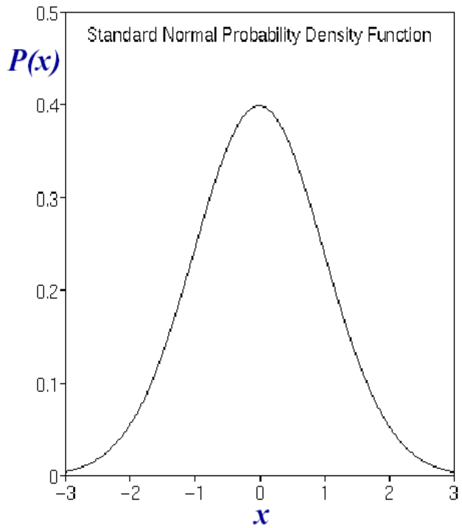


Limiting distribution



$$= \int_A^B f(x)\,dx$$

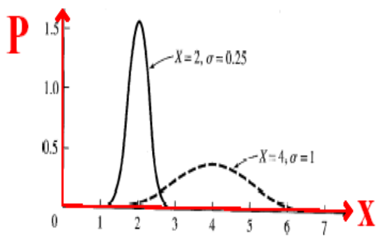
Standard Normal (Gaussian) Distribution



- Many probability distributions, including errors, approach the Normal distribution.
  - Biological parameters
  - Test scores
  - Any combination of random variables (Central Limit Theorem).
- Normal distribution has **average** and **standard deviation** as parameters.

The Normal Distribution

$$P_{X,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

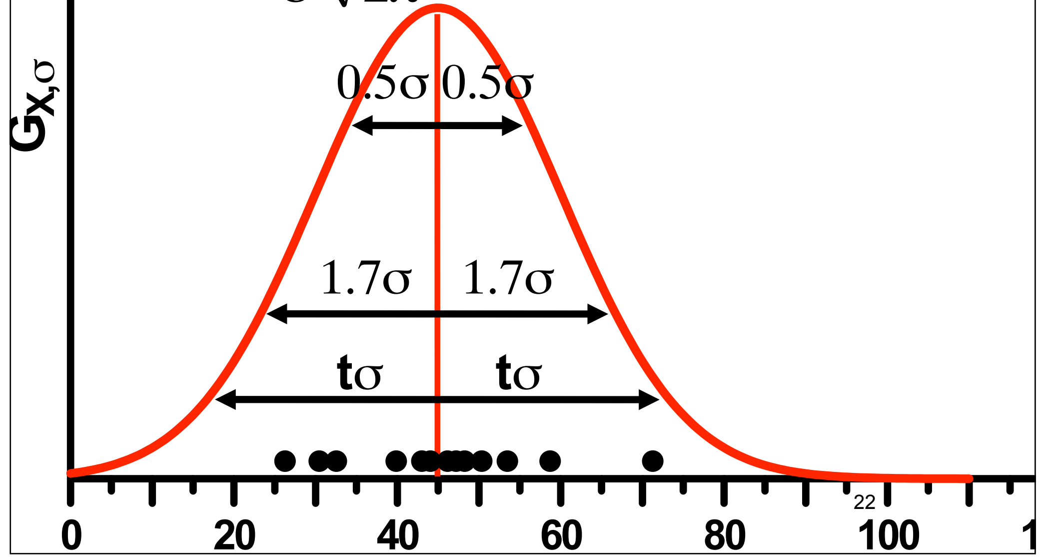


- X and σ are parameters of the Normal distribution.
- X is the true mean of the distribution.
- The RMS width of the distribution is σ.
- x is the independent variable.
- P is the probability density to measure x.

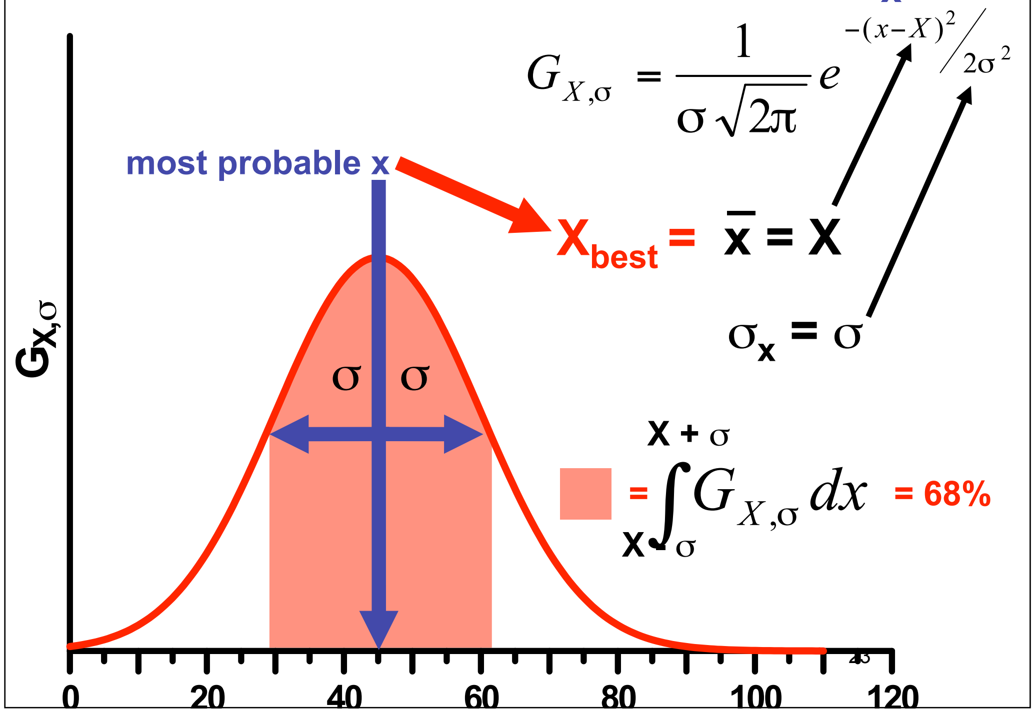
What are the units of $P(x)$?

Integrating Gauss distribution

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$



Gauss distribution and $\bar{x}$, σ_x



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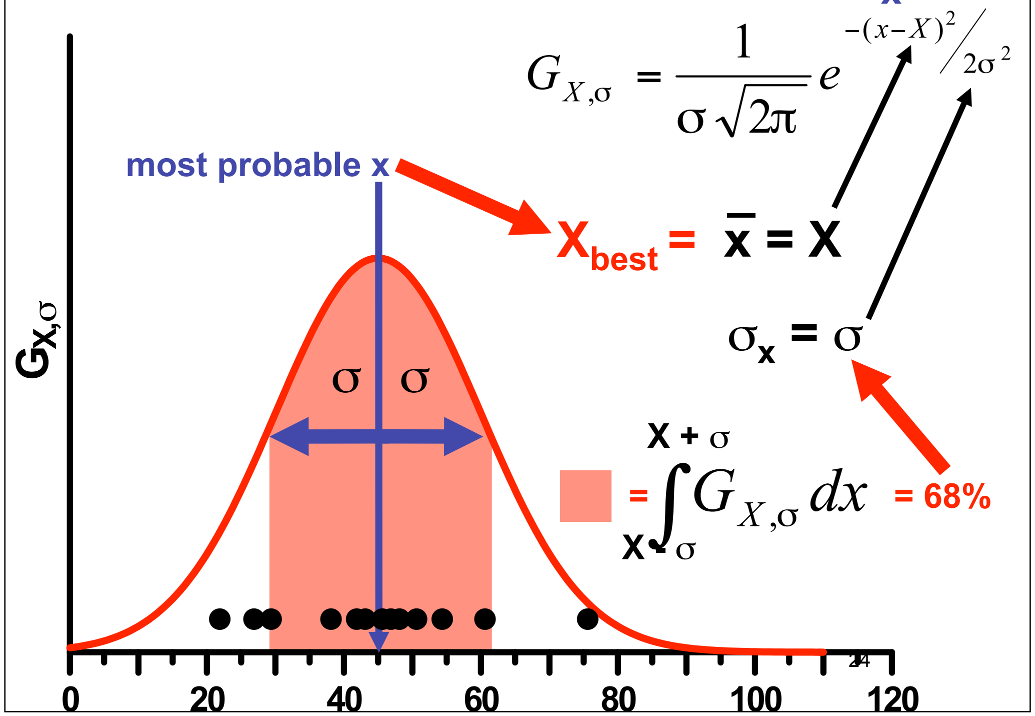
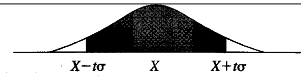


t=1

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$, as a function of t.



t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80
2.3	97.86	97.91	97.97	98.02	98.07	98.12	98.17	98.22	98.27	98.32
2.4	98.36	98.40	98.45	98.49	98.53	98.57	98.61	98.65	98.69	98.72
2.5	98.76	98.79	98.83	98.86	98.89	98.92	98.95	98.98	99.01	99.04
2.6	99.07	99.09	99.12	99.15	99.17	99.20	99.22	99.24	99.26	99.29
2.7	99.31	99.33	99.35	99.37	99.39	99.40	99.42	99.44	99.46	99.47
2.8	99.49	99.50	99.52	99.53	99.55	99.56	99.58	99.59	99.60	99.61

t=1.47

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
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0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
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1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.5[illegible]	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
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2.3	97.86	97.91	97.97	98.02	98.07	98.12	98.17	98.22	98.27	98.32
2.4	98.36	98.40	98.45	98.49	98.53	98.57	98.61	98.65	98.69	98.72
2.5	98.76	98.79	98.83	98.86	98.89	98.92	98.95	98.98	99.01	99.04
2.6	99.07	99.09	99.12	99.15	99.17	99.20	99.22	99.24	99.26	99.29
2.7	99.31	99.33	99.35	99.37	99.39	99.40	99.42	99.44	99.46	99.47
2.8	99.49	99.50	99.52	99.53	99.55	99.56	99.58	99.59	99.60	99.61
2.9	99.63	99.64	99.65	99.66	99.67	99.68	99.69	99.70	99.71	99.72
3.0	99.73									
3.5	99.95									
4.0	99.994									

Limiting Distributions; Normal Distribution (Ch.5)

Plan:

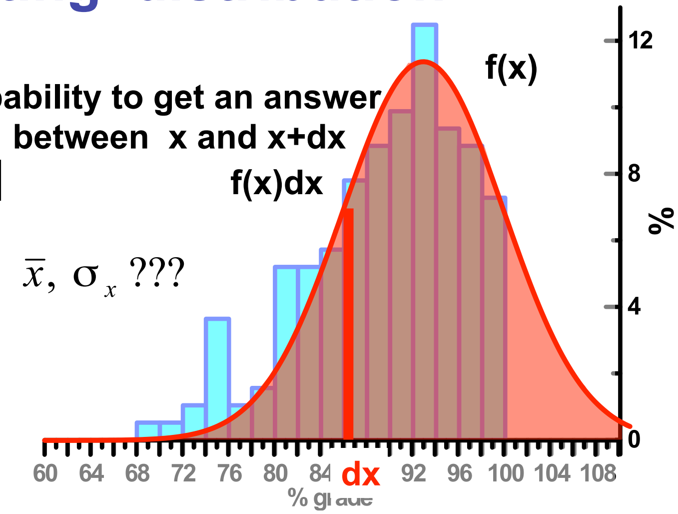
- **Limiting distributions: physical meaning**
- **Normal (Gauss) distribution**
- **Addition in quadrature, SDOM, etc**

Limiting distribution

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

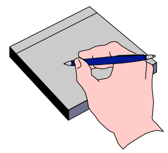
probability to get an answer
between x and x+dx
f(x)dx

$\bar{x}$, σ_x ???



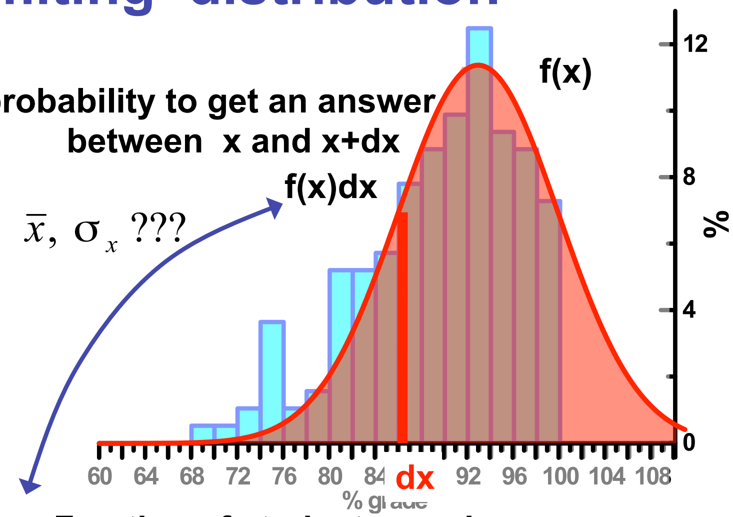
$$\blacksquare = \int_A^B f(x)dx$$

Limiting distribution



$$\bar{x} = \frac{\sum x_i}{N}$$

$\bar{x}, \sigma_x$???



$$\bar{x} = \frac{\sum x_k n_k}{N}$$

score

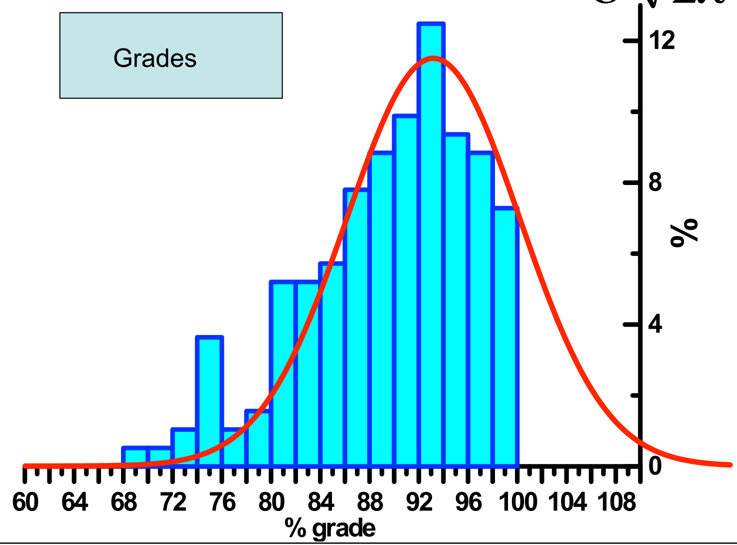
# of times x_k appears

$\frac{n_k}{N}$ → Fraction of students scoring x_k … or probability of x_k → $f(x_k)dx_k$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx \qquad \bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx$$

Gauss distribution and random errors

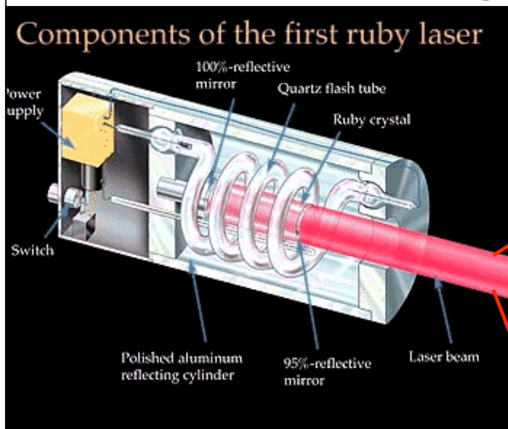
$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

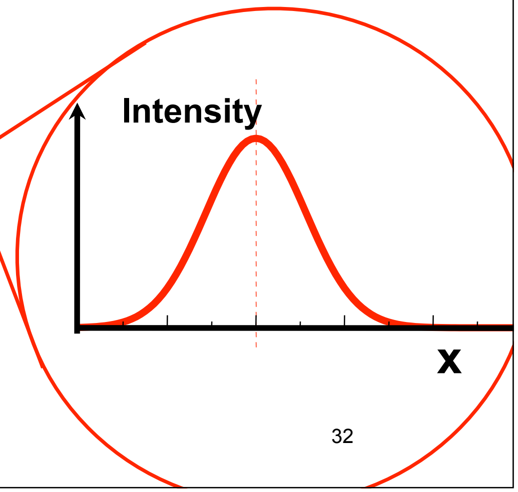



GN1097668Y3
UTSCHE BUNDESBANK
Banknote
ZEHN DEUTSCHE MARK
10
1777–1855 Carl Friedr. Gauß
GN1097668Y3
$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
$\mu-\sigma$
μ
$\mu+\sigma$

Gauss distribution and random errors

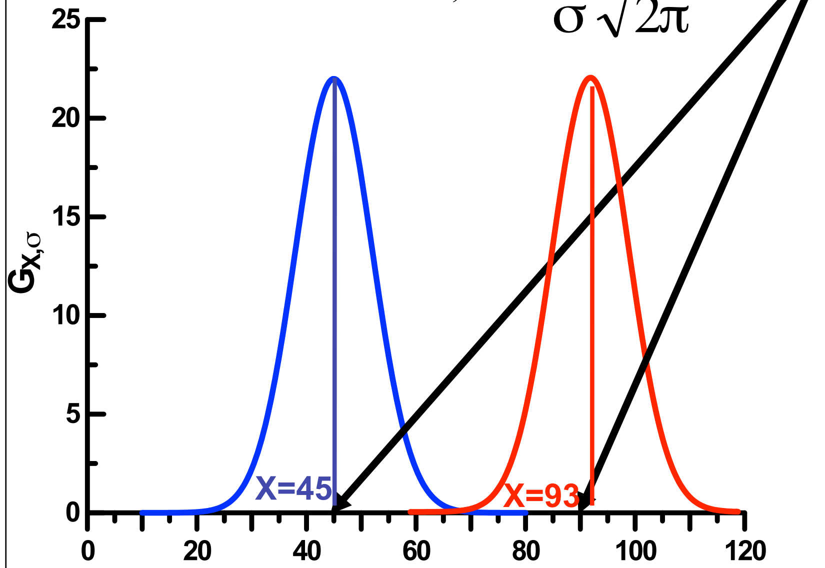
$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$





Normal (Gauss) distribution

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$



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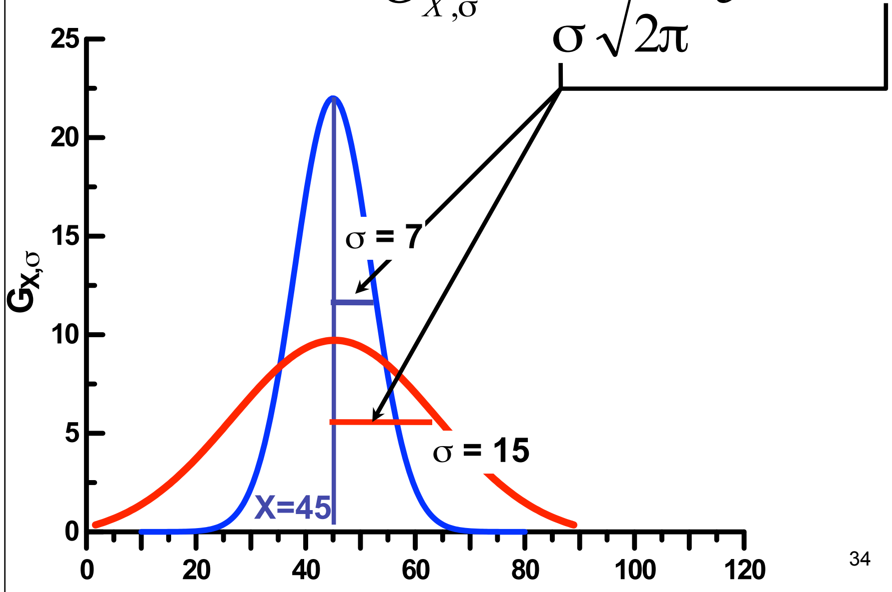
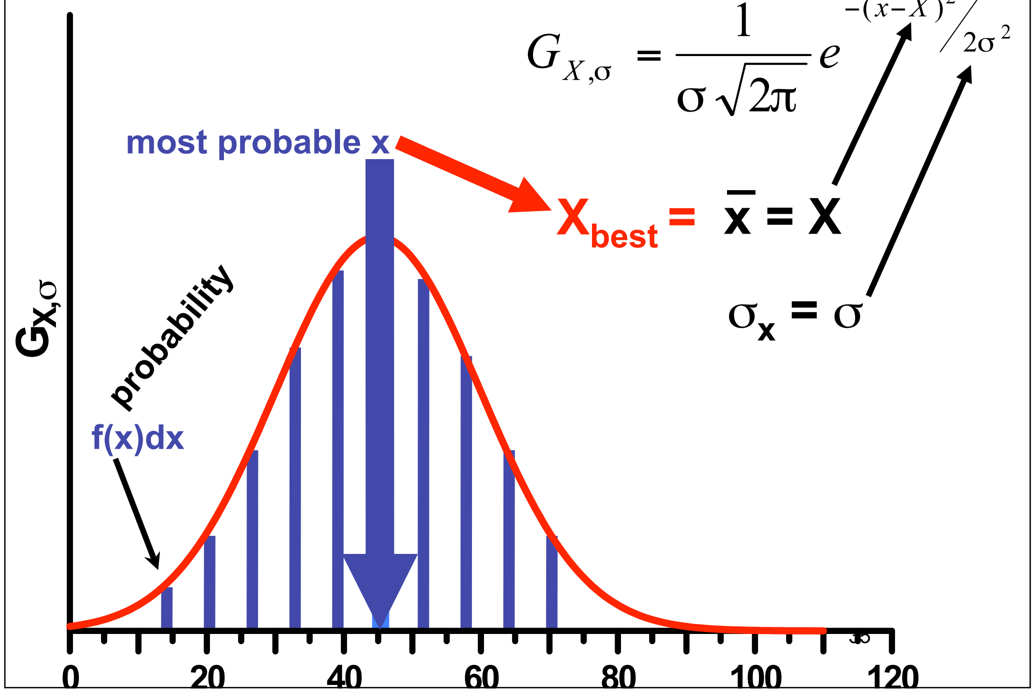


Gauss distribution and $\bar{x}$, σ_x

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$



Physical Constants

$c = 2.997{+}8\ \text{m/s} \approx 1\ \text{ft/ns}$

$c_s = 343\ \text{m/s} \approx 1\ \text{ft/ms}$

$f = c/\lambda$

$\epsilon_0 = 8.85{-}12\ \text{C}^2/\text{N}\cdot\text{m}^2$

$\mu_0 = 4\pi{-}7\ \text{T}\cdot\text{m/A}$

$c^2 = 1/\mu_0\epsilon_0$

$h = 6.626{-}34\ \text{J}\cdot\text{s} = 4.136{-}15\ \text{eV}\cdot\text{s}$

$\mathcal{E} = hf$

$\lambda = 1240\ \text{nm}\left(\frac{\mathcal{E}}{1\text{eV}}\right)^{-1}$

$\lambda_{BB}^{max} = 250\ \text{nm}\left(\frac{kT}{1\text{eV}}\right)^{-1} = 2.9\ \text{nm}\ T^{-1}$

$e = 1.602{-}19\ \text{Coul}$

$V = \frac{e}{4\pi\mathcal{E}_0 r} = 27.2\ \text{Volts}\left(\frac{a_0}{r}\right)$

$1\text{eV} = 1.602{-}19\ \text{Joule} = 11{,}600\ ^\circ\text{K}\ k_B$

$\frac{1}{40}\text{eV} \approx 300^\circ\text{K}$

$\Delta\mathcal{E} = q\,\Delta V$

$N_A = 6.022{+}23$

$m_p = 1.673{-}27\ \text{kg} = 938\ \text{MeV}/c^2$

$\mathcal{E} = mc^2$

$m_e = 9.109{-}31\ \text{kg} = 511\ \text{keV}/c^2 = m_p/1836$

$\lambda_e = 1.227\ \text{nm}\left(\frac{\mathcal{E}_e}{1\text{eV}}\right)^{-1/2}$

$\lambda_m = h/mv = h/\sqrt{2m}\sqrt{\tfrac{1}{2}mv^2}$

$a_0 = \frac{h^2\epsilon_0}{\pi m e^2} = .0529\ \text{nm} = .529\ \text{Å}$

from $\frac{mv_\theta^2}{r} = \frac{e^2}{4\pi\mathcal{E}_0 r^2}$

and $\lambda_e = \frac{h}{mv_\theta} = \frac{2\pi r}{n{=}1}$

or $L_\theta = mv_\theta r = n\,h/2\pi$

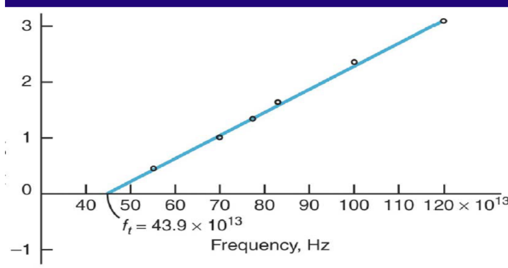
Bulk:

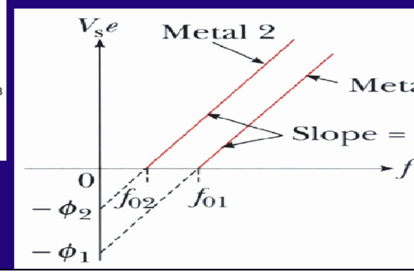
$1\ \text{A}\cdot\text{h} = 3600\ \text{Coul} = 2.3{+}22\,e^- = \frac{N_A}{27}\,e^-$

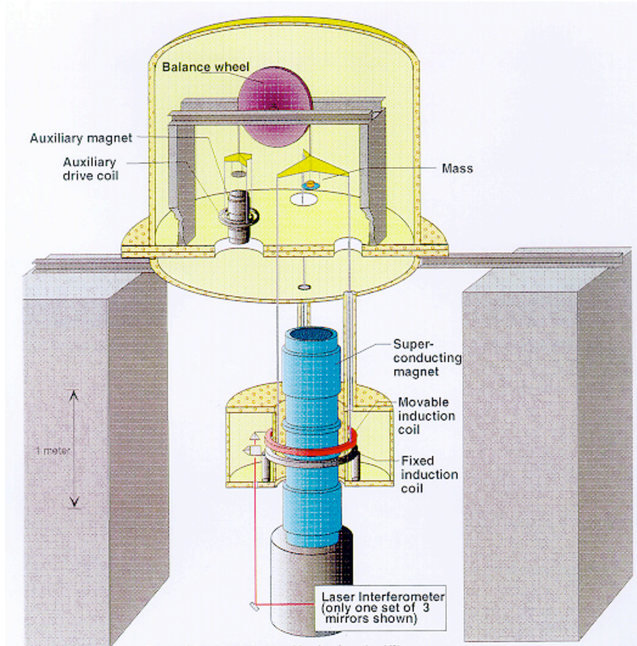
$1\ \text{A}\cdot\text{h}\cdot\text{Volt} = 3600\ \text{Joules} = \frac{N_A}{27}\ \text{eV}$

- Accepted Value of h=6.626 × 10^{-34} J.sec
- You measure freq = 45 THz with uncertainty:
- dF=4.5 THz
- What is best estimate for the Uncertainty in Energy = hF?
- E = 3.0 ± 0.3 × 10^{-20} J

$$V_s e = hf - \phi$$







The NIST Watt Balance: Monitoring the Kilogram
A schematic showing the basic elements of the experiment and the future vacuum enclosure.

$$(K_J)^2 R_K = (2e/h)^2 (h/e^2) = 4/h\,.$$

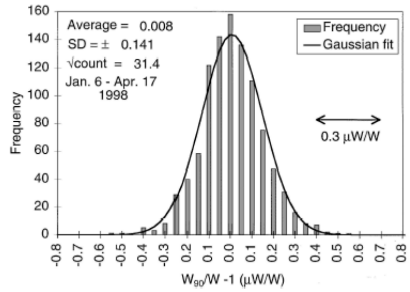


FIG. 2. Histogram of most recent 989 watt measurements.

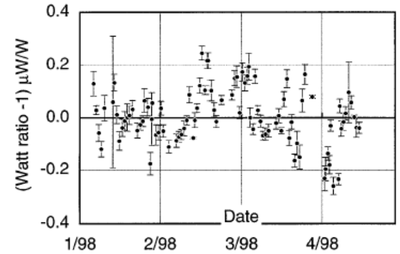
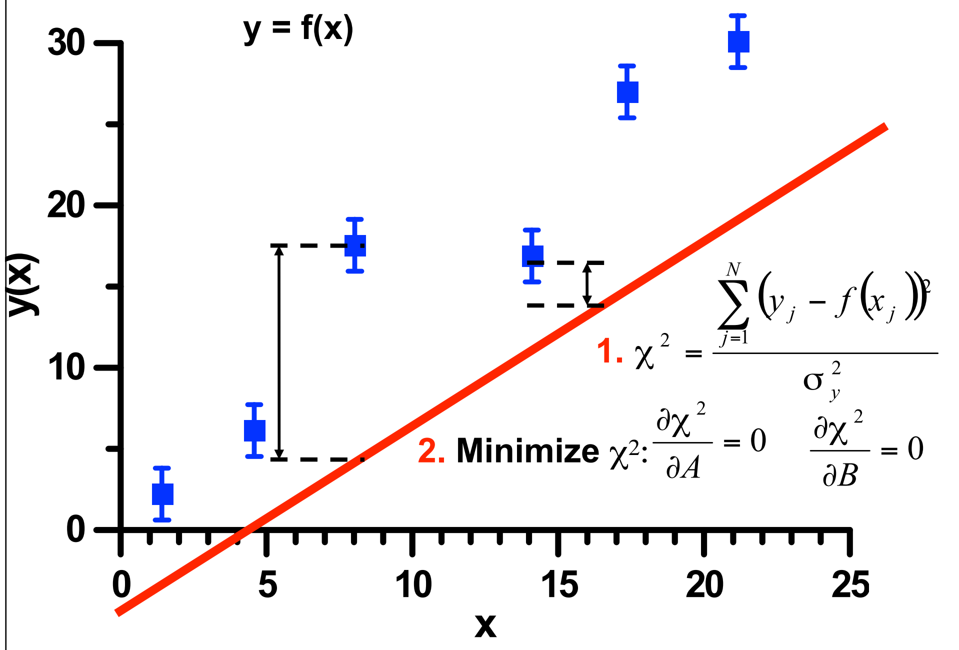


FIG. 3. The daily average of the latest watt results. The error bars are the standard deviation of the mean each day's results.

TABLE II. Relative standard uncertainties in the NIST watt experiment.

Uncertainty source	Value (nW/W)
Reference transfers (type B)	
Mass	20
Resistance	8
Voltage	30
Length	5
Frequency	5
Gravity	7
External effects	
Refractive index	43
Mass buoyancy	23
Alignments	40
Leakage resistance	20
Magnetic flux z-profile fit	20
Knife-edge hysteresis	20
RF noise offsets	10
RSS subtotal	82
Statistical type A	30
Combined	87

Next time: LEAST SQUARES FITTING



Fitting Voltage Data to V=IR

$$\frac{\partial \chi^2}{\partial R} = 0$$

$$IMPLIES:$$

N = number of data points. In this example, N=4

$$R = \frac{\sum_i^N I_i V_i}{\sum_i^N I^2{}_i}$$

What is the Error on the Best-Fit Parameter R?

Our general formula, which always applies, is:

$$\sigma_R = \sqrt{\left(\frac{\partial R}{\partial V_1}\right)^2 \sigma_{v_1}^2 + \left(\frac{\partial R}{\partial V_2}\right)^2 \sigma_{v_2}^2 + \ldots \left(\frac{\partial R}{\partial V_N}\right)^2 \sigma_{v_N}^2}$$

Since:

$$\left(\frac{\partial R}{\partial V_1}\right)^2 = I_1^2, \left(\frac{\partial R}{\partial V_N}\right)^2 = I_N^2$$

$$and: \sigma_{V_N} = 1mV$$

Putting it all together:

$$so: \sigma_R = \frac{1mV\sqrt{\sum_i^N I_i^2}}{\sum_i^N I_i^2}$$

Check units are right, error has same units as R.

LEAST SQUARES FITTING EXAMPLE

current [mA]	voltage [mV]	voltage error [mV]	voltage measured [mV]	voltage uncertainty [mV]	x^2 [mA^2]	xy [mA*mV]	voltage from fit [mV]
1.0	2.0	-0.8	1.2	1.0	1.0	1.2	2.1
2.0	4.0	0.3	4.3	1.0	4.0	8.6	4.2
3.0	6.0	1.0	7.0	1.0	9.0	20.9	6.3
4.0	8.0	0.0	8.0	1.0	16.0	32.1	8.4
this is "x_i"			this is "y_i"	this is "σ_Y"	$\sum x^2$ 30.0	$\sum xy$ 62.9	

This is the true signal

This is the true signal with error (uncertainty).

What we would measure in real-life

Our model: V=I×R

R from Fit: $R = \sum(xy)/\sum(x^2)$ 2.1 Ω

Error in R comes from partial derivative of numerator with respect to y_i only

Error in R $\sigma_R = \sigma_Y/\sqrt{(\sum x^2)}$ 0.2 Ω